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ENTROPIES OF SCALAR FIELDS ON THREE DIMENSIONAL BLACK HOLES

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Abstract

Thermodynamics of scalar fields is investigated in three dimensional black hole backgrounds in two approaches. One is mode expansion and direct computation of the partition sum, and the other is the Euclidean path integral approach. We obtain a number of exact results, for example, mode functions, Hartle-Hawking Green functions on the black holes, Green functions on a cone geometry, free energies and entropies. They constitute reliable bases for the thermodynamics of scalar fields. It is shown that thermodynamic quantities largely depend upon the approach to calculate them, boundary conditions for the scalar fields and regularization method. We find that, in general, the entropies are not proportional to the area of the horizon and that their divergent parts are not necessarily due to the existence of the horizon.

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1 Introduction

Thermodynamics of black holes has been an enigma in theoretical physics. It stands just at the junction of general relativity, quantum mechanics and statistical mechanics. We have thought that its understanding leads to physics beyond that we have at present, in particular, quantum gravity.

However, we have not yet understood the true meaning of thermodynamic laws of black holes, and neither important related problems such as Hawking radiation and quantum coherence.

In respect of these issues, some proposals have been made recently [1]-[5]. For example, (i) the Bekenstein-Hawking entropy and the entropy of a quantum field in a black hole background are the same object, i.e., the response of the Euclidean path integral to the introduction of a conical singularity to the underlying geometry [6, 2]. (ii) The entropy of the quantum field is obtained by tracing over local degrees of freedom inside the horizon (" geometric entropy ") [7, 2], by explicit counting of states [1] or by the Euclidean path integral. (iii) It is proportional to the area of the horizon and gives the first quantum correction to the Bekenstein-Hawking entropy [1, 2]. (iv) Divergences appear due to the blow up of density of states associated to the horizon [8, 1, 2] and they can be removed by the renormalization of the gravitational coupling constant [1]. (v) Consequently, the problem of information loss and that of renormalizability of quantum gravity are intimately related.

Nevertheless, the system of a four dimensional black hole and a scalar field is quite complicated. Thus we have to resort to some approximations and somewhat formal arguments. Moreover, we do not know whether the various approaches for calculating the thermodynamic quantities are equivalent. There is no reason that the equivalence must holds a priori. These prevent us from clear understanding of the arguments.

In this article, we shall work with the three dimensional black holes of Einstein gravity discovered by Bañados, Teitelboim and Zanelli [9, 10]. The three dimensional black holes share many of the features of those in four dimensions, and moreover they provide us with considerably simple systems. Thus we can expect to avoid technical difficulties in four dimensions and to be able to perform explicit analysis of their thermodynamics including matter.

Therefore, we shall mainly pursue two purposes in this article. One is to construct reliable bases for the thermodynamics of quantum scalar fields in the three dimensional black

hole backgrounds. The other is to clarify the validity of the recent arguments explained above by explicit calculations. We believe that our results serve for deep understanding of thermodynamics of black holes.

We organize the rest of this article as follows. First, we briefly review the three dimensional black holes in Sec.2 . Next, in Sec.3 , we study the statistical mechanics of quantum scalar fields by explicit mode expansion and direct computation of the partition sum. We consider two boundary conditions in order to examine the dependence of the thermodynamic quantities upon boundary conditions . One requires the regularity of the scalar fields at the origin. In this case, we obtain exact expressions of the thermodynamic quantities at an arbitrary temperature such as the free energies and the entropies. The other requires the regularity of the scalar fields at the outer horizon with a cutoff [8]. We have explicit forms of the thermodynamic quantities and estimate them in the limit of the vanishing cutoff. In order to examine the equivalence between various approaches to the thermodynamics of quantum fields, we study it also in the Euclidean path integral approach. For this purpose, Sec.4 is devoted to construction of Green functions of scalar fields with arbitrary mass on the three dimensional black holes. It turns out that our construction gives the Green functions defined with respect to the Hartle-Hawking vacuum [11, 12]. By making use of the Euclidean Hartle-Harking Green functions, we investigate the statistical mechanics of the scalar fields in Sec.5 . We obtain exact forms of free energies at the Hawking temperature. Then, we construct the Green functions with arbitrary period with respect to the imaginary time, namely, those on a cone geometry. They enable us to obtain exactly the thermodynamic quantities at an arbitrary temperature. In particular, we calculate the entropies at the Hawking temperature and estimate their divergent parts. Finally, in Sec.6 , conclusions and discussions are given. The construction of Green functions in the universal covering space of three dimensional anti-de Sitter space ($CAdS_3$), which is necessary in Sec.4, is summarized in Appendix.

2 Three Dimensional Black Holes

Let us begin with a brief review of the three dimensional black hole discovered by Bañados et al [9, 10]. The three dimensional black hole is most easily obtained by making use of some identifications under a discrete subgroup of the isometry group of three dimensional anti-de Sitter space (AdS_3).

AdS$_3$ is realized as the three dimensional hyperboloid

$$-u^2 - v^2 + x^2 + y^2 \;=\; -l^2\,, \tag{2.1}$$

in a four dimensional space with the metric

$$ds^2 \;=\; -du^2 - dv^2 + dx^2 + dy^2\,. \tag{2.2}$$

We introduce two parameters r_+, and r_- $(r_+ \geq r_-)$, which turn out shortly to be the radii of the outer and inner horizons of the black hole, and we perform the following transformation into coordinates (t, r, ϕ) :

$$\begin{array}{lll}
\text{Region I.} \quad r_+ < r & : & \begin{array}{ll} u = \sqrt{\tilde{r}^2}\cosh\tilde{\phi}\,, & v = \sqrt{\tilde{r}^2 - l^2}\sinh\tilde{t}\,, \\ x = \sqrt{\tilde{r}^2}\sinh\tilde{\phi}\,, & y = \sqrt{\tilde{r}^2 - l^2}\cosh\tilde{t}\,, \end{array} \\
\text{Region II.} \quad r_- < r < r_+ & : & \begin{array}{ll} u = \sqrt{\tilde{r}^2}\cosh\tilde{\phi}\,, & v = \sqrt{l^2 - \tilde{r}^2}\cosh\tilde{t}\,, \\ x = \sqrt{\tilde{r}^2}\sinh\tilde{\phi}\,, & y = \sqrt{l^2 - \tilde{r}^2}\sinh\tilde{t}\,, \end{array} \\
\text{Region III.} \quad 0 < r < r_- & : & \begin{array}{ll} u = \sqrt{-\tilde{r}^2}\sinh\tilde{\phi}\,, & v = \sqrt{l^2 - \tilde{r}^2}\cosh\tilde{t}\,, \\ x = \sqrt{-\tilde{r}^2}\cosh\tilde{\phi}\,, & y = \sqrt{l^2 - \tilde{r}^2}\sinh\tilde{t}\,, \end{array}
\end{array} \tag{2.3}$$

where

$$\tilde{r}^2 = l^2\left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right)\,, \qquad \begin{pmatrix} \tilde{t} \\ \tilde{\phi} \end{pmatrix} = \frac{1}{l}\begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix}\begin{pmatrix} t/l \\ \phi \end{pmatrix}\,. \tag{2.4}$$

With the above coordinates, the metric becomes

$$\begin{aligned} ds^2 \;&=\; -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2\,, \\ &=\; -\left[\frac{r^2}{l^2} - M\right] dt^2 - J dt d\phi + \left[\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right]^{-1} dr^2 + r^2 d\phi^2\,, \end{aligned} \tag{2.5}$$

with $-\infty < t, \phi < +\infty$. Here

$$N^2 \;=\; \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}\,, \qquad N^\phi \;=\; -\frac{r_+ r_-}{l r^2}\,, \tag{2.6}$$

$$l^2 M \;=\; r_+^2 + r_-^2\,, \qquad lJ \;=\; 2 r_+ r_-\,, \tag{2.7}$$

and M and J are identified with the mass and the angular momentum of the black hole, respectively.

The metric has a Killing vector ∂_ϕ. Then by making the identifications under the discrete subgroup of the isometry group generated by this Killing vector,

$$\phi \;\longrightarrow\; \phi + 2\pi n\,, \quad (n \in \mathbf{Z})\,, \tag{2.8}$$

we get the black hole spacetime.

Note that the scalar curvature is a constant,

$$R = -6l^{-2}\,, \tag{2.9}$$

because the black hole spacetime is locally AdS_3.

In the rest of the present paper, we shall consider quantum scalar fields in the three dimensional black hole backgrounds and their thermodynamics.

3 Statistical Mechanics of Scalar Fields : Partition Sum

In this section, we consider the thermodynamics of scalar fields by mode expansion and direct computation of the partition sum. In this approach, the relation between the entropy and state counting is clear. In order to study dependence of the thermodynamic quantities upon boundary conditions, we consider two cases. In both cases, we require that the scalar fields tend to vanish rapidly enough at spatial infinity. In addition, we impose on them regularity at the origin for one case, or at the horizon for the other case. The former is usually adopted for wave functions, and the latter is adopted in the so-called " brick wall " model [8, 1]. Although it is possible to consider other various boundary conditions, we do not take them because physical meaning is not clear in most cases. It turn out that the thermodynamic quantities largely depend upon the boundary conditions.

3.1 Mode functions

Now let us consider a scalar field with mass squared m^2 in the three dimensional black hole back ground. The field equation is given by

$$(\Box - \mu l^{-2})\psi(x) = 0\,. \tag{3.1}$$

Since $R = -6l^{-2}$, $\mu l^{-2} = m^2$ for a scalar field minimally coupled to the background metric and $\mu l^{-2} = m^2 + (1/8)R = m^2 - (3/4)l^{-2}$ for a conformally coupled scalar field. In terms of the coordinates (t, r, ϕ), the D'Alembertian operator, $\Box$, is given by

$$\begin{aligned} \Box\psi &= \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\right)\psi \\ &= -\frac{1}{r^2N^2}\left[r^2\partial_t^2 - \left(\frac{r^2}{l^2} - M\right)\partial_\phi^2 + J\partial_t\,\partial_\phi\right]\psi + \frac{1}{r}\partial_r\,\left(rN^2\partial_r\psi\right)\,. \end{aligned} \tag{3.2}$$

Changing variables to $v = r^2/l^2$, the field equation becomes as

$$0 = \left[v l^2 \partial_t^2 - (v - M)\partial_\phi^2 + J\partial_t \, \partial_\phi + \mu\Delta(v) \right] \psi - 4\Delta(v)\partial_v \left(\Delta(v)\partial_v \psi \right) , \tag{3.3}$$

where

$$\Delta(v) = \left(v - \frac{r_+^2}{l^2} \right) \left(v - \frac{r_-^2}{l^2} \right) \equiv (v - v_+)(v - v_-) . \tag{3.4}$$

The above equation is solved through separation of variables :

$$\psi_{n\omega} = e^{-i\omega t} \, e^{in\phi} f_{n\omega}(v) , \tag{3.5}$$

where n is an integer. The equation for the radial function is given by

$$f_{n\omega}'' + \frac{\Delta'(v)}{\Delta(v)} f_{n\omega}' + \frac{1}{4\Delta^2(v)} \left\{ n(Mn - J\omega) - \mu\Delta(v) - (n^2 - l^2\omega^2)v \right\} f_{n\omega} = 0 , \tag{3.6}$$

where we have denoted the derivative with respect to v by the prime.

This equation has three regular singular points at $v = v_-, v_+, \infty$ corresponding to the inner horizon, the outer horizon and the spatial infinity, respectively. Thus the solution is given by hypergeometric functions. This is confirmed as follows. First, let $f_{n\omega}$ be of the form

$$f_{n\omega} = (v - v_+)^\alpha (v - v_-)^\beta g_{n\omega} , \tag{3.7}$$

where α and β are purely imaginary numbers defined by

$$\begin{aligned} \alpha^2 &= -\frac{1}{4(v_+ - v_-)^2} \left(r_+\omega - \frac{r_-}{l} n \right)^2 , \\ \beta^2 &= -\frac{1}{4(v_+ - v_-)^2} \left(r_-\omega - \frac{r_+}{l} n \right)^2 . \end{aligned} \tag{3.8}$$

The signs of α and β are irrelevant in the following discussion and we do not bother to specify them. Let us make the change of variables

$$u = \frac{1}{v_+ - v_-} (v - v_-) . \tag{3.9}$$

Then Eq.(3.6) becomes

$$u(1-u) g_{n\omega}''(u) + \{ c - (a + b + 1)u \} \, g_{n\omega}'(u) - a \, b \, g_{n\omega}(u) = 0 , \tag{3.10}$$

where

$$
\begin{aligned}
a &= (\alpha+\beta)+\frac{1}{2}(1+\sqrt{1+\mu})\,, \\
b &= (\alpha+\beta)+\frac{1}{2}(1-\sqrt{1+\mu})\,, \\
c &= 2\beta+1\,.
\end{aligned} \tag{3.11}
$$

This is nothing but the hypergeometric equation, and $g_{n\omega}(u)$ is given by hypergeometric functions.

The hypergeometric equation has two independent solutions around each regular singular point. Thus we must impose boundary conditions to specify the solution. First, we consider the behavior of $f_{n\omega}(v)$ as $v \to \infty \quad (u \to \infty)$. Near the infinity, we have two independent solutions :

$$
f^{1,\infty}_{n\omega}(v) = (v-v_+)^{\alpha}(v-v_-)^{\beta}u^{-a}F(a, a-c+1; a-b+1; 1/u)\,, \tag{3.12}
$$

$$
f^{2,\infty}_{n\omega}(v) = (v-v_+)^{\alpha}(v-v_-)^{\beta}u^{-b}F(b, b-c+1; b-a+1; 1/u)\,, \tag{3.13}
$$

where F is the hypergeometric function. From (3.11), we find that $f^{2,\infty}_{n\omega}$ becomes divergent as $v \to \infty$ for $\mu > 0$, while $f^{1,\infty}_{n\omega}$ comes to vanish for arbitrary μ.

The authors of [13]-[15] have discussed the quantization of scalar fields in anti-de Sitter space or its covering space, which has timelike spatial infinity and needs special boundary conditions there. If we require the condition to conserve energy following them, the surface integral of the energy momentum tensor $\lim_{r\to\infty}\int dS_i\sqrt{-g}T^i{}_t$ must vanish. This means $\sqrt{r}f_{n\omega} \to 0 \quad (r \to \infty)$, and only $f^{1,\infty}_{n\omega}$ satisfies this condition. Therefore we concentrate on $f^{1,\infty}_{n\omega}$ and drop the superscript $(1,\infty)$ in the following.

3.2 Case I : regularity at the origin

In this section, we impose on $f_{n\omega}(v)$ regularity at the origin ($r=0$) as is usual for radial functions, and study the thermodynamics under this boundary condition. Introducing appropriate cutoffs, we obtain exact results.

It is easy to see that $f_{n\omega}$ is regular at the origin because $r=0$ corresponds to none of $z=0,1,\infty$. Thus we have no restriction on the value of ω. Then we proceed to calculate thermodynamic quantities. First, we consider the case of $J \neq 0$. Recall that the system of the rotating black hole and the scalar field has a chemical potential Ω_H. This is the angular velocity of the outer horizon:

$$
\Omega_H = \frac{d\phi}{dt}\Big|_{r=r_+} = -N^{\phi}\Big|_{r=r_+} = \frac{r_-}{lr_+}\,. \tag{3.14}
$$

In addition, the system has superradiant scattering modes given by the condition

$$\omega - \Omega_H \, n \leq 0 \,, \tag{3.15}$$

where ω and n are the energy and angular momentum of the scalar field, respectively. Thus we have to regularize the (grand) partition function by introducing the cutoff N_1 for the occupation number of the particle for each mode satisfying (3.15).

With these remarks in mind, we get the partition function for a single mode labeled by ω and n,

$$\begin{aligned} Z_o(\beta;\omega,n) &= \sum_{m=0}^{\infty} e^{-m(\omega-\Omega_H n)} \\ &= \begin{cases} \left(1- \; e^{-\beta(\omega-\Omega_H n)}\right)^{-1} & \text{for} \quad \omega-\Omega_H n>0 \\ N_1 & \text{for} \quad \omega-\Omega_H n=0 \\ \frac{1- \; e^{-N_1\beta(\omega-\Omega_H n)}}{1- \; e^{-\beta(\omega-\Omega_H n)}} & \text{for} \quad \omega-\Omega_H n<0 \end{cases} \,. \end{aligned} \tag{3.16}$$

Then we obtain the total partition function,

$$Z_o(\beta) \;=\; \prod_{\omega,n} Z_o(\beta;\omega,n)\,, \tag{3.17}$$

and the free energy,

$$\begin{aligned} -\beta F_o(\beta) &= \sum_{\omega,n} \ln \; Z_o(\beta;\omega,n) \\ &= -\sum_{|n|=0}^{N_2} \frac{1}{s}\int_0^{\infty} d\omega \ln\left(1- \; e^{-\beta(\omega-\Omega_H n)}\right) + \sum_{n=0}^{N_2} N_1 \\ &\qquad + \sum_{n=0}^{N_2} \frac{1}{s}\int_0^{n\Omega_H} d\omega \ln\left(1- \; e^{-\beta N_1(\omega-\Omega_H n)}\right) \,, \end{aligned} \tag{3.18}$$

where N_2 is the cutoff for the absolute value of quantum number n, and s is the minimum spacing of ω. Note that s^{-1} is the density of states and the above result is divergent as $s \to 0$ regardless of the existence of the horizon. By making the change of variables $t = \beta(\omega - \Omega_H n)$ for the first term and $t = N_1\beta(n\Omega_H - \omega)$ for the third term , we obtain

$$\begin{aligned} -\beta F_o(\beta) &= \frac{1}{s}\Bigg[\frac{\pi^2}{6\beta}(2N_2+1) + \frac{\beta}{12}\Omega_H^2(N_1-1)N_2(N_2+1)(2N_2+1) \\ &\qquad + \frac{1}{N_1\beta}\sum_{n=1}^{N_2}\int_0^{N_1\beta\Omega_H n} dt \;\ln\left(1- \; e^{-t}\right)\Bigg] + N_1(N_2+1)\,. \end{aligned} \tag{3.19}$$

In the limit $N_1 \to \infty$, the last term in the bracket is simplified to be $-N_2\zeta(2)/(N_1\beta)$.

Since entropy is given by

$$S(\beta) \;=\; \beta^2 \frac{\partial F}{\partial \beta}\,, \tag{3.20}$$

we get

$$\begin{aligned} S_o(\beta) \;=\; & \frac{1}{s}\left[\frac{\pi^2}{3\beta}(2N_2+1) - \Omega_H \sum_{n=1}^{N_2} n\ \ln\left(1-\ e^{-N_1\beta\Omega_H n}\right)\right. \\ & \left. + \frac{2}{N_1\beta}\sum_{n=1}^{N_2}\int_0^{N_1\beta\Omega_H n} dt\ \ln\left(1-\ e^{-t}\right)\right] + N_1(N_2+1)\,. \end{aligned} \tag{3.21}$$

For the $J=0$ case, the chemical potential vanishes, and the partition function for a single mode is given by

$$\begin{aligned} Z_o(\beta;\omega,n) \;&=\; \sum_{m=0}^{\infty}\ e^{-m\omega} \\ &=\; \begin{cases} \left(1-\ e^{-\beta\omega}\right)^{-1} & \text{for} \quad \omega>0 \\ N_1 & \text{for} \quad \omega=0 \end{cases}\,. \end{aligned} \tag{3.22}$$

Then the free energy becomes

$$\begin{aligned} -\beta F_o(\beta) \;&=\; \sum_{\omega,n} \ln\ Z_o(\beta;\omega,n) \\ &=\; \frac{(2N_2\!+\!1)\pi^2}{6s\beta} + N_1(2N_2+1)\,. \end{aligned} \tag{3.23}$$

Finally, we get the entropy ;

$$S_o(\beta) \;=\; \frac{(2N_2\!+\!1)\pi^2}{3s\beta} + N_1(2N_2+1)\,. \tag{3.24}$$

From the expressions of the entropies (3.21) and (3.24), we find that the entropies are not proportional to the area of the outer horizon ($2\pi r_+$) and that their divergences are not due to the existence of the outer horizon.

3.3 Case II : regularity at the outer horizon

Since the redshift factor of the black hole becomes divergent at the horizon, one may expect that something singular occurs there. Indeed, we find that $f_{n\omega}$ becomes singular. Thus another natural boundary condition is to require regularity at the outer horizon [8, 1]. In this section, we introduce a cutoff for the distance from the outer horizon in order to regulate $f_{n\omega}$. Then we impose the regularity at the outer horizon on the scalar

fields, and study the thermodynamics under this condition. At present, we do not know whether this is the unique boundary condition which is physically acceptable, and leave this as an open problem.

First, let us study the behavior of $f_{n\omega}$ near the outer horizon ($r = r_+$, i.e., $u = 1$). By making use of a linear transformation formula with respect to the hypergeometric function, we get

$$\begin{aligned} f_{n\omega}(v) \ \propto\ & (u-1)^{\alpha} u^{\beta} F(a, b; 2\alpha+1; 1-u) \\ & + \Theta\ (u-1)^{-\alpha} u^{-\beta} F(1-b, 1-a; -2\alpha+1; 1-u)\,, \end{aligned} \tag{3.25}$$

where

$$\Theta = \frac{\Gamma(1-b)\Gamma(c-b)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)\Gamma(c-a-b)}\,. \tag{3.26}$$

From $(\Gamma(z))^* = \Gamma(z^*)$ and Eq.(3.11), we see that $|\Theta| = 1$. Thus we can set

$$\Theta = -\, e^{-2\pi i\theta_0} \qquad (0 \le \theta_0 < 1)\,, \tag{3.27}$$

where θ_0 is determined by ω and n through a, b and c. Thus by choosing an appropriate normalization constant, it follows that

$$\begin{aligned} f_{n\omega}(v) = & (u-1)^{\alpha} u^{\beta}\ e^{\pi i\theta_0} F(a, b; 2\alpha+1; 1-u) \\ & -(u-1)^{-\alpha} u^{-\beta}\ e^{-\pi i\theta_0}\ F(1-b, 1-a; -2\alpha+1; 1-u)\,. \end{aligned} \tag{3.28}$$

Then by introducing an infinitesimal constant, ϵ_H, and substituting $u = 1 + \epsilon_H^2/l^2$ ($\epsilon_H^2 \propto r^2 - r_+^2$) into $f_{n\omega}$, we get the behavior of $f_{n\omega}(v)$ near the outer horizon as follows

$$f_{n\omega} \ \xrightarrow{\epsilon_H \to 0}\ e^{\alpha \ln(\epsilon_H^2/l^2) + \pi i\theta_0} - e^{-(\alpha \ln(\epsilon_H^2/l^2) + \pi i\theta_0)}\,. \tag{3.29}$$

Clearly, the radial function is singular at the outer horizon. Therefore we require [1] that $f_{n\omega}$ vanishes at $u = 1 + \epsilon_H^2/l^2$ instead of $u = 1$.

Since α is purely imaginary, the condition is easily realized. From (3.8), we get

$$\omega = \Omega_H\, n + C(k + \theta_0)\,, \qquad (k \in \mathbf{Z})\,, \tag{3.30}$$

$$C = \frac{2(v_+ - v_-)\pi}{r_+ \ln(l^2/\epsilon_H^2)}\,. \tag{3.31}$$

[1] In [1], the Dirichlet boundary condition at the horizon is imposed on a scalar field. This is given by $K_{i\omega}(\xi\epsilon) = 0$ where $K_{i\omega}$ is the modified Bessel function, ξ is a function of the momentum and the mass, $\xi = \sqrt{\vec{k}^2 + m^2}$, and ϵ is essentially the same as ϵ_H. Supposed that $\xi\epsilon << 1$, then the boundary condition is solved by expanding $K_{i\omega}$ to be $\omega \ln(\xi\epsilon/2) \sim k\pi$. This confirms the result by the WKB method adopted in [1] in the leading order of ϵ.

This shows that ω and θ_0 are labeled by two integers k and n, i.e., $\omega = \omega(k,n)$, $\theta_0 = \theta_0(k,n)$. Note that C^{-1} becomes singular as $\epsilon_H \to 0$.

Let us consider thermodynamic quantities for $J \neq 0$ first. They are obtained in the same way as in the previous section. The partition function for a single mode labeled by k and n is given by

$$
\begin{aligned}
Z_h(\beta; k, n) &= \sum_{m=0}^{\infty} e^{-m(\omega - \Omega_H n)} \\
&= \begin{cases} \left(1 - e^{-\beta(\omega-\Omega_H n)}\right)^{-1} & \text{for} \quad k+\theta_0 > 0 \\ N_1 & \text{for} \quad k+\theta_0 = 0 \\ \frac{1 - e^{-N_1\beta(\omega-\Omega_H n)}}{1 - e^{-\beta(\omega-\Omega_H n)}} & \text{for} \quad k+\theta_0 < 0 \end{cases} .
\end{aligned} \tag{3.32}
$$

Hence we get the total partition function,

$$
Z_h(\beta) = \prod_{k,n} Z_h(\beta; k, n) . \tag{3.33}
$$

and the free energy,

$$
\begin{aligned}
-\beta F_h(\beta) &= \sum_{k,n} \ln \; Z_h(\beta; k, n) \\
&= -\sum_{|n|=0}^{N_2} \sum_{\substack{k+\theta_0 \neq 0 \\ C(k+\theta_0) \geq -\Omega_H n}} \ln\left(1 - e^{-C\beta(k+\theta_0)}\right) + N_1 \sum_{n=0}^{N_2} \delta_{\theta_0(0,n),0} \\
&\quad + \sum_{n=1}^{N_2} \sum_{0 > C(k+\theta_0) \geq -\Omega_H n} \ln\left(1 - e^{-N_1 C\beta(k+\theta_0)}\right) .
\end{aligned} \tag{3.34}
$$

Since $C << 1$ in the limit $\epsilon_H \to 0$, the summation with respect to k can be approximated by integrals. First, note that

$$
\frac{dk}{d\omega} = \frac{1}{C} - \frac{d\theta_0}{dk}\frac{dk}{d\omega} \sim \frac{1}{C} . \tag{3.35}
$$

This shows that the density of states diverges due to the existence of the outer horizon i.e., as $\epsilon_H \to 0$. Then in the same way as in the previous section, we get

$$
-\beta F_h(\beta) \sim \frac{r_+ l^2 s \ln(l^2/\epsilon_H^2)}{2\pi d_H^2} \{-\beta F_o(\beta) - N_1(N_2+1)\} + N_1 \sum_{n=0}^{N_2} \delta_{\theta_0(0,n),0} , \tag{3.36}
$$

where

$$
d_H^2 = r_+^2 - r_-^2 . \tag{3.37}
$$

As for the entropy, we have

$$S_h(\beta) \ \sim \ \frac{r_+ l^2 s \ln(l^2/\epsilon_H^2)}{2\pi d_H^2} \{S_o(\beta) - N_1(N_2+1)\} + N_1 \sum_{n=0}^{N_2} \delta_{\theta_0(0,n),0} \, . \tag{3.38}$$

In the case of $J = 0$, the calculation is performed in the same way, and we obtain

$$-\beta F_h(\beta) \ \sim \ \frac{\pi r_+ l^2 \ln(l^2/\epsilon_H^2)}{12 d_H^2 \beta}(2N_2+1) + N_1 \sum_{|n|=0}^{N_2} \delta_{\theta_0(0,n),0} \, , \tag{3.39}$$

$$S_h(\beta) \ \sim \ \frac{\pi r_+ l^2 \ln(l^2/\epsilon_H^2)}{6 d_H^2 \beta}(2N_2+1) + N_1 \sum_{|n|=0}^{N_2} \delta_{\theta_0(0,n),0} \, . \tag{3.40}$$

From the expressions of the entropies (3.38) and (3.40), we find (i) that the leading terms of the entropies as $\epsilon_H \to 0$ are proportional to r_+ , but there can exist the terms which is not proportional to r_+, (ii) that the entropies diverges due to the outer horizon as $\epsilon_H \to 0$ like $\ln(l^2/\epsilon_H^2)$.

4 Green Functions on the Three Dimensional Black Holes

In the preceding section, we calculated thermodynamic quantities by the straightforward mode sum. In the following, we shall investigate thermodynamics of scalar fields in the Euclidean path integral approach in order to examine the equivalence between the various approaches. For this purpose, we construct Green functions of scalar fields on the black hole and identify the vacuum state in this section.

4.1 Construction of green functions

Quantization of a scalar field in the universal covering space of n-dimensional anti-de Sitter space (CAdS$_n$) is discussed in [13]-[16], and the Feynman Green function is given in terms of hypergeometric functions [16]. In the three dimensional case ($n = 3$), by using a mathematical formula for hypergeometric functions, we get fairly simple form of the Green function,

$$-iG_F(x, x') \ = \ -iG_F(z) \ \equiv \ \frac{1}{4\pi l}(z^2-1)^{-1/2}\left[z + (z^2-1)^{1/2}\right]^{1-\lambda} \, , \tag{4.1}$$

where

$$z \ = \ 1 + l^{-2}\sigma(x, x') + i\varepsilon \, , \tag{4.2}$$

$$\lambda \ = \ \begin{cases} \lambda_\pm \equiv 1 \pm \sqrt{1+\mu} & \text{for} \ \ 0 > \mu > -1 \\ \lambda_+ & \text{for} \ \ \mu \geq 0 \, , \mu = -1 \end{cases} \, , \tag{4.3}$$

(two λ's are possible for $0 > \mu > -1$). $\sigma(x,x')$ is half of the distance between x and x' in the four dimensional embedding space,

$$\sigma(x,x') \;=\; \frac{1}{2}\eta_{\mu\nu}(\xi-\xi')^{\mu}(\xi-\xi')^{\nu}\,, \tag{4.4}$$

where $\eta_{\mu\nu} = diag(-1,-1,+1,+1)$ and ξ and ξ' are the coordinates in the embedding space. Since the derivation of this result is somewhat lengthy and technical, we relegate it to Appendix.

By making use of the above result, Green functions in the three dimensional black hole background are obtained by the method of images [17]-[19] ;

$$\begin{aligned} -iG_{BH}(x,x') &= -i\sum_{n=-\infty}^{\infty} G_F(x,x'_n) \\ &= \frac{1}{4\pi l}\sum_{n=-\infty}^{\infty}(z_n^2-1)^{-1/2}\left[z_n+(z_n^2-1)^{1/2}\right]^{1-\lambda}\,, \end{aligned} \tag{4.5}$$

where

$$x_n \;\equiv\; x\,\Big|_{\phi'\to\phi'-2n\pi}\,, \qquad z_n(x,x') \;=\; z(x,x'_n)\,. \tag{4.6}$$

By using (2.3), we have

$$\begin{aligned} z_n(x,x')-i\varepsilon \;=\; \frac{1}{d_H^2}\Bigg[&\sqrt{r^2-r_-^2}\sqrt{r'^2-r_-^2}\cosh\left(\frac{r_-}{l^2}\Delta t-\frac{r_+}{l}\Delta\phi_n\right) \\ &-\sqrt{r^2-r_+^2}\sqrt{r'^2-r_+^2}\cosh\left(\frac{r_+}{l^2}\Delta t-\frac{r_-}{l}\Delta\phi_n\right)\Bigg]\,, \end{aligned} \tag{4.7}$$

where

$$\Delta t \;=\; t-t'\,, \qquad \Delta\phi_n \;=\; \phi-\phi'+2n\pi\,. \tag{4.8}$$

Note that in the case of the conformally coupled massless scalar field, namely $\mu = -3/4$, $\lambda_\pm = 3/2, 1/2$ holds and the corresponding Green functions become

$$-iG_{BH}(x,x') \;=\; \frac{1}{2^{\lambda+1}\pi l}\sum_{n=-\infty}^{\infty}\left[\frac{1}{\sqrt{z_n-1}}\pm\frac{1}{\sqrt{z_n+1}}\right]. \tag{4.9}$$

The above results coincide with those of [17]-[19] constructed from the Green functions with the "Neumann" or "Dirichlet" boundary conditions in $CAdS_3$.

4.2 Boundary conditions and the vacuum

In the previous section, we have constructed Green functions on three dimensional black holes. However, the physical meaning of the Green functions is not clear unless we specify its boundary conditions and identify the vacuum with respect to which they are defined.

It turns out that G_{BH} which we have constructed satisfies the boundary conditions : (i) to be regular at infinity (ii) to be analytic in the lower half plane on the future complexified outer horizon (iii) to be analytic in the upper half plane on the past complexified outer horizon. These conditions fix G_{BH} as a solution of the inhomogeneous wave equation [11]. This means that G_{BH} is regarded as the Green function constructed by the Kruskal modes, namely as the Hartle-Hawking Green function. In other words, the vacuum with respect to which G_{BH} is defined is identified with the Hartle-Hawking vacuum [11, 12].

Now let us prove the above statement. For brevity, we concentrate only on the case $r, r' \geq r_+$ in the following. It is easy to see that the boundary condition (i) is satisfied from the definition of G_{BH}.

The conditions (ii) and (iii) have already been verified for the case of the conformally coupled massless scalar field on the non-rotating black hole (i.e. $J = 0$) [18]. Thus we follow the strategy of [18].

First, we introduce Kruskal coordinates [10] by

$$\begin{aligned} V &= R(r)\, e^{a_H t}, \qquad U \;=\; -R(r)\, e^{-a_H t}, \\ R(r) &= \sqrt{\left(\frac{r - r_+}{r + r_+}\right)\left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}}, \end{aligned} \tag{4.10}$$

where

$$a_H = \frac{r_+^2 - r_-^2}{r_+ l^2}. \tag{4.11}$$

In these coordinates, the metric becomes

$$ds^2 = \Omega^2(r) dU dV + r^2 \left(N^\phi dt + d\phi\right)^2, \tag{4.12}$$

$$\Omega^2(r) = \frac{(r^2 - r_-^2)(r + r_+)^2}{a_H^2 r^2 l^2} \left(\frac{r - r_-}{r + r_+}\right)^{r_-/r_+}. \tag{4.13}$$

Second, let us recall the Kerr black hole. In this case, we have to introduce an angle coordinate rotating with the outer horizon in order to obtain the expression of the metric

regular on the outer horizon and to extend the spacetime maximally [20]. In the same way, we introduce a new angle coordinate rotating with the outer horizon,

$$\phi^+ \;=\; \phi - \Omega_H t\,, \tag{4.14}$$

because we are interested in the situation just on the outer horizon. Note that $N^\phi dt + d\phi = d\phi^+$ on the outer horizon. In this coordinate, it follows that

$$\begin{aligned} z_n(x,x') - i\varepsilon \;=\; & \frac{1}{d_H^2}\left[\sqrt{r^2-r_-^2}\sqrt{r'^2-r_-^2}\cosh\left(\frac{r_+}{l}\Delta\phi_n^+\right)\right. \\ & \left. -\sqrt{r^2-r_+^2}\sqrt{r'^2-r_+^2}\cosh\left(a_H\Delta t - \frac{r_-}{l}\Delta\phi_n^+\right)\right]\,, \end{aligned} \tag{4.15}$$

where

$$\Delta\phi_n^+ \;=\; = \phi^+ - \phi'^+ + 2n\pi\,. \tag{4.16}$$

Then let us examine the analyticity of the Green functions on the past complexified outer horizon given by $V = 0$ and $\mathrm{Re}(-U) > 0$. In terms of (t,r), this condition is equivalent to

$$\begin{cases} r \longrightarrow r_+ \\ t \longrightarrow -\infty\,, \end{cases} \quad \text{with} \quad \sqrt{r-r_+}\; e^{-a_H t} \longrightarrow \sqrt{l}A\,, \tag{4.17}$$

where A is a constant determined by the value of U and with the property Re $A > 0$. It follows from (4.10) that Im $A > 0$ and Im $A < 0$ correspond to the lower and the upper half planes of U, respectively.

In the limit (4.17), z_n becomes

$$\begin{aligned} z_n(x,x') - i\varepsilon \;\longrightarrow\; & \frac{1}{d_H^2}\left[\sqrt{r_+2-r_-^2}\sqrt{r'^2-r_-^2}\cosh\left(\frac{r_+}{l}\Delta\phi_n^+\right)\right. \\ & \left. -\frac{1}{2}\sqrt{2r_+ l}\sqrt{r'^2-r_+^2}\; e^{a_H t' + (r_-/l)\Delta\phi_n^+}\; A\right]\,. \end{aligned} \tag{4.18}$$

Recalling the form of G_{BH}, i.e. (4.5), we see that each component in the summation in G_{BH} has singularities where $z_n = \pm 1$. From $r' > r_+$ and (4.18), we see that the solutions to $z_n = \pm 1$ on the past complexified outer horizon are of the form

$$A \;=\; \alpha_0 + i\varepsilon\,, \tag{4.19}$$

for both signs, where α_0 is some positive number. Thus we find that the each component in G_{BH} is regular in the upper half plane of U.

Then we shall use Weierstrass's theorem [21] : if a series with analytic terms converges uniformly on every compact subset of a region, then the sum is analytic in that region, and the series can be differentiated term by term. In fact, we can check that the series in G_{BH} is converges uniformly. Thus we conclude that G_{BH} is analytic in the upper half plan of the past complexified outer horizon.

The proof of the analyticity on the lower half plan of the future complexified outer horizon is much the same. Thus we omit it for brevity.

Finally, we make a comment. The Hartle-Hawking Green function on a black hole was originally defined in the path-integral formalism as a generalization of the Feynman Green function in Minkowski spacetime [11]. In our case, G_F is also defined so as to conform to the Feynman Green function in the flat limit (see (A .25) and the comment below it). Thus it is natural that G_F satisfies the Hartle-Hawking boundary condition.

5 Statistical Mechanics of Scalar Fields by Hartle -Hawking Green Functions

In the previous section, we see that the Green function G_{BH} is the Hartle-Hawking Green function, which is often used for discussion on thermodynamics of black holes and Hawking radiation. In this section, we discuss statistical mechanics of a scalar field by using G_{BH}. In the Euclidean path integral approach to statistical field system, thermodynamic quantities are obtained by making use of Euclidean Green functions.

5.1 Euclidean green functions

Let us define the Green function on the Euclidean black hole geometry. Introducing the Euclidean time $\tau = it$ and the " Euclidean " angle $\varphi = -i\phi$ for $J \neq 0$ and $\varphi = \phi$ for $J = 0$, this is given by

$$G^E_{BH}(\Delta\tau, \Delta\varphi; r, r') \equiv \sum_{n=-\infty}^{\infty} G^E_F(\Delta\tau, \Delta\varphi_n; r, r')\,, \tag{5.1}$$

$$G^E_F(\Delta\tau, \Delta\varphi; r, r') \equiv \begin{cases} iG_F(\Delta t, \Delta\phi; r, r')|_{\substack{\Delta t = i\Delta\tau \\ \Delta\varphi = \Delta\phi}} & \text{for} \quad J = 0 \\ \\ iG_F(\Delta t, \Delta\phi; r, r')|_{\substack{\Delta t = i\Delta\tau \\ \Delta\varphi = -i\Delta\phi}} & \text{for} \quad J \neq 0 \end{cases}\,. \tag{5.2}$$

Here $\Delta\tau$ and $\Delta\varphi_n$ is defined as (4.8), and the superscript E means Euclidean quantities. The factors in front of G_F are chosen so that the physical quantities calculated later will have real values with appropriate signs.

Since $(\Box - \mu l^{-2}) G_{BH} = (1/\sqrt{-g})\delta(x-x')$, we have

$$(\Box^E - \mu l^{-2}) G^E_{BH} \;=\; \begin{cases} -\frac{1}{\sqrt{|g^E|}}\,\delta^E(x-x') & \text{for } J=0 \\[2ex] \frac{i}{\sqrt{|g^E|}}\,\delta^E(x-x') & \text{for } J\neq 0 \end{cases}\,, \tag{5.3}$$

where g^E is defined by the line element

$$ds^2_E \;=\; \begin{cases} N^2 d\tau^2 + r^2 d\varphi^2 + N^{-2}dr^2 & \text{for } J=0 \\[2ex] N^2 d\tau^2 - r^2(N^\phi d\tau - d\varphi)^2 + N^{-2}dr^2 & \text{for } J\neq 0 \end{cases}\,. \tag{5.4}$$

Please note that the metric is not positive definite for $J \neq 0$.

Then let us consider thermal properties of G^E_{BH}. For a moment, we concentrate on the case of $J \neq 0$. A thermal Green function at temperature β^{-1} and with a chemical potential ν conjugate to angular momentum is defined by

$$G^E_\beta(x,x';\nu) \;=\; \mathrm{tr}\,\left[\, e^{-\beta(\hat{H}-\nu\hat{L})}\; T\left(\psi(x)\psi(x')\right)\right] \Big/ \;\mathrm{tr}\;\left[\, e^{-\beta(\hat{H}-\nu\hat{L})}\right]\,, \tag{5.5}$$

where T denotes the (Euclidean) time ordered product and $\hat{H}$ and $\hat{L}$ are the generators of time translation and rotation, respectively. From the above definition,

$$G^E_\beta(\tau,\varphi,r;\tau',\varphi',r';\nu) \;=\; G^E_\beta(\tau+\beta,\varphi-\nu\beta,r;\tau',\varphi',r';\nu)\,. \tag{5.6}$$

Because the Green function G_{BH} is a function of z_n, from (4.7) we find that G^E_{BH} is periodic under

$$\begin{aligned} \delta\left(\frac{r_-}{l^2}\,\tau + \frac{r_+}{l}\,\varphi\right) &= 2\pi m \\ \delta\left(\frac{r_+}{l^2}\,\tau + \frac{r_-}{l}\,\varphi\right) &= 2\pi n \qquad (m,n\in\mathbf{Z})\,, \end{aligned} \tag{5.7}$$

where $\delta(...)$ means the variation of the arguments. Thus G^E_{BH} is of double period

$$\begin{pmatrix} \delta(\tau/l) \\ \delta\varphi \end{pmatrix} \;=\; \frac{2\pi l}{d_H^2}\begin{pmatrix} -r_- & r_+ \\ r_+ & -r_- \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}\,. \tag{5.8}$$

If we require that, as $J\to 0$ $(r_-\to 0)$, the chemical potential vanishes, the fundamental period is determined uniquely as

$$\tau \;\to\; \tau + \frac{2\pi}{a_H}\,n\,, \qquad \varphi \to \varphi - \frac{2\pi}{a_H l}\,\nu\,, \tag{5.9}$$

$$\nu \;=\; \frac{r_-}{l r_+} = \Omega_H\,. \tag{5.10}$$

It is easy to see that this result is valid also in the case of $J = 0$. Therefore, we conclude that G^E_{BH} can be regarded as a thermal Green function at the inverse temperature

$$\beta_H \ \equiv \ 2\pi/a_H \, , \tag{5.11}$$

and with the chemical potential Ω_H. We shall calculate thermodynamic quantities by making use of G^E_{BH}. In the following, we shall explicitly show the period of the Green functions, for example, as $G^E_{BH}(\Delta\tau, \Delta\varphi, r, r'; \beta_H)$.

It is instructive to consider the behavior of the metric near the outer horizon. Let us introduce a coordinate η by

$$r \ = \ r_+ + \frac{2}{a_H}\eta^2 \, . \tag{5.12}$$

Then for small η , the metric becomes

$$ds^2 \ \sim \ -a_H^2 \, \eta^2 \, dt^2 + d\eta^2 + r_+^2 (d\phi^+)^2 \, . \tag{5.13}$$

Moreover, in terms of the Euclidean time $\tau = -it$, (τ, r) represents a plane with the origin $r = r_+$. Therefore we find that β_H is nothing but the period around the outer horizon of the Euclidean black hole, while Ω_H is the angular velocity of the outer horizon (see (3.14)). Thus our result confirms and gives an explicit example to the arguments in the literature of thermodynamics of black holes. In addition, it may be worth noting that for small $\eta, \eta', \Delta\tau, \Delta\varphi$,

$$2\sigma(x, x') \ \sim \ (\Delta\eta)^2 + r_+^2 (\Delta\phi^+)^2 \, . \tag{5.14}$$

Thus the distance of the embedding space becomes that with respect to the metric (5.13).

5.2 Free energy

In this section, we shall calculate the free energy $F(\beta)$, which is given by

$$\beta F(\beta) \ = \ -\frac{1}{2} \, \mathrm{tr} \, \log G^E_{BH}(\beta) \, , \tag{5.15}$$

where the trace is defined by

$$\begin{aligned} \mathrm{tr} \, (\, ... \,) \ &= \ \int d^3x \sqrt{|g^E|} \, \lim_{x \to x'} (\, ... \,) \\ &= \begin{cases} \int_0^\beta d\tau \int_0^{2\pi} d\varphi \int_{r_+}^\infty dr \cdot r \ \lim_{x \to x'} (\, ... \,) & \text{for } J = 0 \\ \\ \int_0^\beta d\tau \int_0^{\Omega_H \beta} d\varphi \int_{r_+}^\infty dr \cdot r \ \lim_{x \to x'} (\, ... \,) & \text{for } J \neq 0 \end{cases} \, . \end{aligned} \tag{5.16}$$

In (5.16), we have set the lower end of the integration with respect to r to be r_+. The reasons are twofold; (i) in the Euclidean geometry, the topology of (τ, r) space is $\mathbf{R}^2$ and the origin corresponds to $r = r_+$, and (ii) it turns out that the entropy becomes complex if we perform integration below r_+.

From the expressions (5.3) and (5.15), it follows that

$$\frac{\partial}{\partial\mu}\left(\beta\ F(\beta)\right) \quad = \quad -\frac{1}{2l^2}\ \mathrm{tr}\ G^E_{BH}(\beta)\,. \tag{5.17}$$

In the case of flat spacetime, the expression like (5.15) is divergent. Thus we have to regularize it by the expression like (5.17). For getting the right answer, we then integrate out (5.17). However since the derivation from (5.15) to (5.17) is rather formal, the final result may depend upon with which of these we start. Indeed, we shall find that the result depends on the choice. Here we start with (5.17) according to the prescription in the case of flat spacetime.

We consider the case $J \neq 0$ first. In this case, we have

$$\frac{\partial}{\partial\mu}\left(\beta\ F(\beta)\right)\Big|_{\beta=\beta_H} \quad = \quad -i\ \frac{\Omega_H}{4l^2}\beta^2 \sum_{n=-\infty}^{\infty}\int_{r_+^2}^{\infty} d(r^2) \lim_{r\to r'} G_F(z_n^0;\beta_H)\ \Big|_{\beta=\beta_H}, \tag{5.18}$$

where $z_n^0 = z_n \Big|_{\Delta\tau=\Delta\varphi=0}$ and we have used the fact that the integrand is independent of τ and φ.

Recalling the expression of G_F and z_n, i.e., (4.1) and (4.7), we see that the integrand with $n = 0$ in the summation diverges in the limit $r \to r'$. Thus we remove this term for a moment.

It is useful to notice that $G_F(z_n;\beta_H)$ and z_n^0 are written as

$$-iG_F(z_n;\beta_H) \quad = \quad \begin{cases} \frac{l^{-1}}{4\pi}\frac{1}{1-\lambda}\frac{d}{dz_n}\ e^{(1-\lambda)\coth^{-1} z_n} & \text{for } \lambda \neq 1 \\[2ex] \frac{l^{-1}}{4\pi}\left(z_n^2-1\right)^{-1/2} & \text{for } \lambda = 1 \end{cases}\ , \tag{5.19}$$

$$z_n^0\Big|_{r=r'} \quad = \quad \frac{1}{d_H^2}\left\{\left(r^2 - r_-^2\right)\ c_n^+\ - \left(r^2 - r_+^2\right)\ c_n^-\ \right\}, \tag{5.20}$$

where

$$c_n{}^{\pm} \quad = \quad \cosh\left(2\pi n\frac{r_\pm}{l}\right)\,. \tag{5.21}$$

Since the infinitesimal imaginary part of z_n is irrelevant for the discussion, we have omitted it. We shall do it also in the following unless it is necessary. Then by making the change

of variables from r^2 to z_n^0, we get

$$
\begin{aligned}
&-i\int_{r_+^2}^{\infty} d(r^2)\lim_{r\to r'} G_F(z_n^0;\beta_H) \\
&= \begin{cases} \frac{l^{-1}}{4\pi}\frac{d_H^2}{(\ c_n^+\ -\ c_n^-\ )(1-\lambda)}\left(z+\sqrt{z^2-1}\right)^{1-\lambda}\Big|_{c_n^+}^{\infty} & \text{for}\lambda\neq 1 \\ \frac{l^{-1}}{4\pi}\frac{2d_H^2}{(\ c_n^+\ -\ c_n^-\ )}\log\left(z+\sqrt{z^2-1}\right)\Big|_{c_n^+}^{\infty} & \text{for}\lambda=1 \end{cases} .
\end{aligned} \tag{5.22}
$$

Therefore the integral diverges at the upper end for $\lambda<1$ (i.e., $\lambda=\lambda_-$) and for $\lambda=1$ (i.e., $\lambda=\lambda_+\,,\ \mu=-1$), while for $\lambda>1$ (i.e., $\lambda=\lambda_+,\mu\neq-1$) we get

$$
\begin{aligned}
&\frac{\partial}{\partial\mu}\left(\beta\ F(\beta)\right)\Big|_{\beta=\beta_H} \\
&\qquad = \frac{l^{-3}}{8\pi}\Omega_H\beta_H^2\frac{d_H^2}{\lambda-1}\sum_{n\geq1}^{\infty}\frac{1}{c_n^+\ -\ c_n^-}\ e^{-2\pi(\lambda-1)nr_+/l}\ +C_0\,,
\end{aligned} \tag{5.23}
$$

where C_0 is the divergent term coming from $n=0$. By integrating the above expression, we obtain

$$
\begin{aligned}
\beta\ F(\beta)\Big|_{\beta=\beta_H} &= -\frac{l^{-2}\Omega_H}{8\pi^2 r_+}\beta_H^2 d_H^2\sum_{n\geq1}^{\infty}\frac{1}{n(\ c_n^+\ -\ c_n^-\ )}\ e^{-2\pi(\lambda-1)nr_+/l} \\
&\qquad + \ \text{const.} \qquad\qquad (\text{for }\lambda>1)\,.
\end{aligned} \tag{5.24}
$$

For the case of $J=0$, the calculation is performed in a similar way, and the result is

$$
\begin{aligned}
\beta\ F(\beta)\Big|_{\beta=\beta_H} &= -\frac{d_H^2\beta_H}{4\pi r_+ l^2}\sum_{n\geq1}^{\infty}\frac{1}{n(\ c_n^+\ -\ c_n^-\ )}\ e^{-2\pi(\lambda-1)nr_+/l} \\
&\qquad + \ \text{const.} \qquad\qquad (\text{for }\lambda>1)\,.
\end{aligned} \tag{5.25}
$$

5.3 Green functions on a cone geometry

In order to calculate the entropy, we have to differentiate the Euclidean Green function with respect to β with the chemical potential fixed. Thus we need Green functions with period different from $\beta_H=2\pi/a_H$ with Ω_H fixed. Namely, we need to construct Green functions on $\tau-r$ plane with a deficit angle around the origin, i.e., on a cone geometry. For this purpose, we first regard $\Delta\tau$ and $\Delta\varphi^+$ $(=\Delta\varphi+\Omega_H\Delta\tau)$ as independent variables, and then we fix the value of $\Delta\varphi^+$. After that, we construct the Green function with an arbitrary period β with respect to $\Delta\tau$. By this procedure, it is assured that the chemical potential is unchanged.

Long time ago, the problem of constructing solutions of certain differential equations with period different from 2π from the one with the period 2π was discussed [22, 23].

Then this method was applied to field theory on curved spaces [24, 25, 5]. We can also make use of this method to obtain the Green function with an arbitrary period β.

We introduce a new variable w by $w = a_H\tau = -ia_H t$ and denote $G_F^E(z_n;\beta_H)$ with $\Delta\varphi_n^+$, r and r' fixed by

$$\tilde{G}_F^E(w-w_n';2\pi) \equiv G_F^E(\Delta\tau,\Delta\varphi_n;r,r';\beta_H)\Big|_{\Delta\varphi^+,r,r':\ \text{fixed}}, \tag{5.26}$$

where

$$w_n = w - i\frac{r_-}{l}\Delta\phi_n^+ . \tag{5.27}$$

Note that $\tilde{G}_F^E$ depends upon $w-w_n'$ through $z_n(x,x')$ as

$$\begin{aligned} z_n(x,x') - i\varepsilon &= \frac{1}{d_H^2}\left[\sqrt{r^2-r_-^2}\sqrt{r'^2-r_-^2}\cosh\left(\frac{r_+}{l}\Delta\phi_n^+\right)\right. \\ &\qquad \left. -\sqrt{r^2-r_+^2}\sqrt{r'^2-r_+'^2}\cosh\left(i(w-w_n')\right)\right] . \end{aligned} \tag{5.28}$$

Then we shall study the location of singularities of $\tilde{G}_F^E$. This is necessary for writing out the expression of the Green function with an arbitrary period $2\pi\beta/\beta_H$, i.e., $\tilde{G}_F^E(w-w';2\pi\beta/\beta_H)$. In the same way as $G_F(z_n;\beta_H)$, $\tilde{G}_F^E$ has singularities at $z_n=\pm1$. From (5.28), we find that $\tilde{G}_F^E$ has four singularities in the region $-\pi < \text{Re}\,(w-w_n') \leq \pi$. They are located infinitesimally close to the imaginary axis of $w-w_n'$ plane and symmetrically with respect to the point $w-w_n'=0$. In the limit $\Delta\varphi_n^+\to 0$ and $r\to r'$, two of these singularities approach to the same point $w-w_n'=0$.

Now we are ready to construct $\tilde{G}_F^E(w-w_n';2\pi\beta/\beta_H)$. This is given by the Sommerfeld integral representation [22, 23],

$$\tilde{G}_F^E(w-w_n';2\pi\beta/\beta_H) = \frac{\beta_H}{2\pi\beta}\int_\Gamma d\zeta\,\tilde{G}_F^E(\zeta-w_n';2\pi)\frac{e^{i\beta_H\zeta/\beta}}{e^{i\beta_H\zeta/\beta} - e^{i\beta_H w/\beta}} , \tag{5.29}$$

where the contour Γ of the integral is given by the solid line in Fig.1 . This contour consists of two parts and divide the four singularities into two pairs. In the case of $\Delta\varphi_n^+=0$ and $r=r'$, we cannot take such a contour because two of the singularities degenerate into $\zeta-w_n'=0$. Therefore we define $\tilde{G}_F^E$ in this case by

$$\tilde{G}_F^E(w-w_n';2\pi\beta/\beta_H)\Big|_{\substack{\Delta\varphi_n^+=0,\\ r=r'}} \equiv \lim_{\substack{\Delta\varphi_n^+\to0,\\ r\to r'}} \tilde{G}_F^E(w-w_n';2\pi\beta/\beta_H)\Big|_{\substack{\Delta\varphi_n^+\neq0\\ \text{or }r\neq r'}} . \tag{5.30}$$

From $\tilde{G}^E_F(w - w'_n; 2\pi\beta/\beta_H)$, we can get the Green functions with an arbitrary period β on the cone geometry, namely, $G^E_{BH}(x - x'; \beta)$ in a similar way as (4.5) :

$$G^E_{BH}(x - x'; \beta) = \sum_{n=-\infty}^{\infty} G^E_F(x - x'_n; \beta) , \tag{5.31}$$

$$G^E_F(x - x'_n; \beta) = \tilde{G}^E_F(w - w'_n; 2\pi\beta/\beta_H) . \tag{5.32}$$

It is instructive to consider some special cases before we prove that the expression above is actually correct. First, in the case of $\beta = \beta_H/q$, $(q = 1, 2, ...)$, the contour Γ is deformed into Γ' given by the dashed line in Fig.1 . Since the integrand is of period 2π, the contributions from the path made up of straight lines cancel with each other. Thus only the residues inside the circular path contribute to the integral. Therefore we get

$$\tilde{G}^E_F(w - w'_n; 2\pi/q) = \sum_k \tilde{G}^E_F(w(k) - w'_n; 2\pi) , \tag{5.33}$$

where $w(k)$ and $k(\in \mathbf{Z})$ are given by $w(k) = w + 2\pi k/q$ and $-\pi < w(k) \leq \pi$. In this case, the method of images works and we can explicitly check the periodicity. Clearly, in the case of $q = 1$, the r.h.s. of (5.33) reproduces $\tilde{G}^E_F(w - w'_n; 2\pi)$.

Next, let us consider the case $\beta \to \infty$. In the limit $\beta \to \infty$, it follows that

$$\tilde{G}^E_F(w - w'_n : \infty) = \frac{1}{2\pi i} \int_\Gamma \tilde{G}^E_F(w - w'_n; 2\pi) \frac{d\zeta}{\zeta - w} . \tag{5.34}$$

By using this and a formula $\lim_{n\to\infty} \sum_{k=-n}^{n} 1/(x + k) = \pi \cot \pi x$, we obtain another expression for $\tilde{G}^E_F(w - w'_n; \beta_H)$:

$$\begin{aligned} \tilde{G}^E_F(w - w'_n; 2\pi\beta/\beta_H) &= \sum_{k=-\infty}^{\infty} \tilde{G}^E_F(w - w'_n + 2\pi k\beta/\beta_H; \infty) \\ &= \frac{\beta_H}{4\pi i\beta} \int_\Gamma d\zeta \tilde{G}^E_F(\zeta - w'_n; 2\pi) \cot \left\{ \frac{\beta_H}{2\beta}(\zeta - w) \right\} . \end{aligned} \tag{5.35}$$

The equivalence to the former expression is easily checked by noting $\tilde{G}^E_F(w - w'_n; \beta_H) = \tilde{G}^E_F(w - w'_n; -\beta_H)$.

Now we shall check properties of Green functions. First, it is clear that $\tilde{G}^E_F(w - w'_n; 2\pi\beta/\beta_H)$ actually converges because $\tilde{G}^E_F(\zeta - w'_n; 2\pi\beta/\beta_H)$ comes to vanish exponentially as $|\mathrm{Im}\ \zeta| \to \infty$. It is also easy to see that $\tilde{G}^E_F(w - w'_n; 2\pi\beta/\beta_H)$ is of period $2\pi\beta/\beta_H$ by making the change of variables $\zeta - w'_n = \zeta'$.

Finally, let us check that $G^E_F(x - x'; \beta)$ satisfies the inhomogeneous equation. As an example, we consider the case of $J = 0$. From $(\Box^E_{x'} - \mu l^{-2}) G^E_F(x - x'; \beta_H) = -(1/\sqrt{|g^E|})$

$\times\delta^E_{\beta_H}(x-x')$, it follows that

$$(\Box^E_{x'} - \mu l^{-2}) G^E_F(x-x';\infty) \;=\; -\frac{1}{\sqrt{|g^E|}}\,\delta^E_\infty(x-x')\,, \tag{5.36}$$

where we have explicitly denoted the period of the delta function. Thus from the fact $G^E_F(x-x'_n;\beta) = \sum_{k=-\infty}^{\infty} G^E_F(x-x'_n;\infty)\Big|_{\Delta\tau\to\Delta\tau+k\beta}$, we get the desired result :

$$(\Box^E_{x'} - \mu l^{-2}) G^E_F(x-x';\beta) \;=\; -\frac{1}{\sqrt{|g^E|}}\,\delta^E_\beta(x-x')\,. \tag{5.37}$$

For calculating entropy in the later section, let us calculate the derivative of the Green functions. From the cotangent form (5.35), we get the derivative of $G^E_F(x-x'_n;\beta)$ with respect to β ,

$$\begin{aligned}\frac{\partial}{\partial\beta} G^E_F(x-x'_n;\beta) &= -\frac{1}{\beta} G^E_F(x-x'_n;\beta) \\ &\quad + \frac{\beta_H^2}{8\pi i\beta^3}\int_\Gamma d\zeta\; \tilde{G}^E_F(\zeta-w'_n;2\pi)(\zeta-w)\mathrm{cosec}^2\left\{\frac{\beta_H}{2\beta}(\zeta-w)\right\}.\end{aligned} \tag{5.38}$$

In the case of $\beta=\beta_H$, the above expression is fairly simplified. First, we deform the contour Γ into Γ'. In this case, the singularity within the circular path is only at $\zeta=w$, and the contribution from the residue of this singularity cancels with the first term in (5.38). Thus by changing variables, we get

$$\frac{\partial}{\partial\beta} G^E_F(x-x'_n;\beta) \;=\; \frac{1}{4\beta_H}\int_{-\infty}^{\infty} d\zeta'\;\frac{\tilde{G}^E_F(i\zeta'-\pi;\beta_H)}{\cos^2\{(i\zeta'+w'_n-w)/2\}}\,. \tag{5.39}$$

Note that $\tilde{G}^E_F(i\zeta'-\pi;\beta_H)$ is a function of

$$z(\zeta') \;\equiv\; z_n(x,x')\Big|_{w-w'_n=i\zeta'-\pi} = A_n + B\cosh\zeta'\,, \tag{5.40}$$

where

$$A_n \;=\; \frac{\sqrt{r^2-r_-^2}\sqrt{r'^2-r_-^2}}{d_H^2}\cosh\left(\frac{r_+}{l}\Delta\phi_n^+\right)\,, \qquad B \;=\; \frac{\sqrt{r^2-r_+^2}\sqrt{r'^2-r_+^2}}{d_H^2}\,. \tag{5.41}$$

Thus we have

$$\frac{dz}{d\zeta'} \;=\; B\sinh\zeta' \;=\; \pm\sqrt{(z-A_n)^2-B^2}\,, \tag{5.42}$$

and

$$\cos^2\left\{\frac{1}{2}(i\zeta'+w'_n-w)\right\} \;=\; \frac{1}{2}\left\{\; c_n\,\frac{z-A_n}{B} \pm\; s_n\,\frac{\sqrt{(z-A_n)^2-B^2}}{B} \;+\; 1\right\}. \tag{5.43}$$

c_n and s_n are given by

$$c_n = \cosh\left(i(w-w'_n)\right), \qquad s_n = \sinh\left(i(w-w'_n)\right). \tag{5.44}$$

Therefore by making the further change of variables from ζ' to z, we get the fairly simple form :

$$\begin{aligned}&\frac{\partial}{\partial\beta}G_F^E(x-x'_n;\beta)\Big|_{\beta=\beta_H}\\ =\ & -\frac{B}{\beta_H}\int_{A_n+B}^{\infty}dz\ G_F^E(z;\beta_H)\frac{1}{\sqrt{(z-A_n)^2-B^2}}\frac{c_n\ (z-A_n)+B}{(z-A_n+\ c_n\ B)^2}\,.\end{aligned} \tag{5.45}$$

Note that $B>0$ for $r>r_+$ and $B<0$ for $r<r_+$ hold . Thus A_n+B can be less than unity for $r<r_+$, and the Green function $-iG_F^E(z;\beta_H)$ becomes complex. Moreover, it turns out that the entropy also becomes complex due to the contribution from this region. This indicates that we should consider only the region $r>r_+$ in calculation of thermodynamic quantities as we have so far done.

5.4 Entropy

Now we are ready to calculate the entropy. First, we consider the case of $J\neq 0$. From equations like Eq.(5.18) and Eq.(5.45), it follows that

$$\begin{aligned}\frac{\partial}{\partial\mu}S(\beta)\Big|_{\beta=\beta_H} &= -\frac{i}{4l^2}\Omega_H\beta_H^2\sum_{n=-\infty}^{\infty}\int_{r_+^2}^{\infty}d(r^2)\ \lim_{r\to r'}\Big[G_F^E(x-x'_n;\beta_H)\\ &-B\int_{A_n+B}^{\infty}dz\ G_F^E(z;\beta_H)\ \frac{1}{\sqrt{(z-A_n)^2-B^2}}\frac{c_n\ (z-A_n)+B}{(z-A_n+\ c_n\ B)^2}\Big]\Big|_{\Delta\tau=\Delta\varphi=0}\,.\end{aligned} \tag{5.46}$$

The first term is nothing but $\partial_\mu(\beta F(\beta_H))$. Thus by integrating the above expression, we get

$$\begin{aligned}S(\beta_H) &= \beta_H F(\beta_H)+\frac{\Omega_H}{8\pi l^3}\ \beta_H^2\sum_{n=-\infty}^{\infty}\int_{r_+^2}^{\infty}d(r^2)\ \lim_{r\to r'}\Big[B\int_{A_n+B}^{\infty}dz\\ &\times\frac{X^{1-\lambda}\left\{1+(\lambda-1)\log X\right\}}{\log^2 X\sqrt{z^2-1}}\frac{1}{\sqrt{(z-A_n)^2-B^2}}\frac{c_n\ (z-A_n)+B}{(z-A_n+\ c_n\ B)^2}\Big]\Big|_{\Delta\tau=\Delta\varphi=0}+c\,,\end{aligned} \tag{5.47}$$

where

$$X = z+\sqrt{z^2-1}\,, \tag{5.48}$$

and c is a constant independent of μ and is ignored in the flat case.

For the case of $J = 0$, the calculation is modified due to the difference in the definition of the trace. However we can get the entropy in the same way as

$$S(\beta_H) = \frac{1}{4l^3}\,\beta_H \sum_{n=-\infty}^{\infty} \int_{r_+}^{\infty} d(r^2)\ \lim_{r\to r'} \left[B \int_{A_n+B}^{\infty} dz \right. \tag{5.49}$$

$$\left. \times \frac{X^{1-\lambda}\,\{1+(\lambda-1)\log X\}}{\log^2 X\sqrt{z^2-1}} \frac{1}{\sqrt{(z-A_n)^2-B^2}} \frac{c_n\ (z-A_n)-B}{(z-A_n+\ c_n\ B)^2} \right] \Big|_{\Delta\tau=\Delta\varphi=0} + c\,.$$

As we have exact expressions of the entropies, we can study the structure of their divergences without any ambiguity. At present, it is believed that the divergence of the entropy of quantum fields on black holes is closely related to important physical problems [1]-[5]. As divergent parts in the entropy for $J = 0$ can be easily obtained from those for $J \neq 0$, we concentrate on the latter case first.

It is possible that the integral of the second term in (5.47) diverges by the contribution from the region of large r. However, we do not know whether it occurs unless we know the behavior of the integrand in the second term for large r. Thus we leave this as an open question. This possible divergence is regarded as an infrared divergence.

Next, let us consider divergences which come from short distances. For this purpose, we introduce an infinitesimal variable ρ and an infinitesimal constant s by

$$\rho^2 \ = \ r^2 - r_+^2\,, \qquad s^2 \ = r'^2 - r^2\,. \tag{5.50}$$

In the limit $\rho,\ s \longrightarrow 0$, we have

$$A_n \ \sim \ c_n^+ \left(1 + \frac{\rho^2 + s^2/2}{d_H^2}\right)\,, \qquad B = \left(\frac{\rho}{d_H^2}\right)\sqrt{\rho^2+s^2}\,, \tag{5.51}$$

$$\begin{aligned} z^0 \ &\equiv \ z(x,x')\Big|_{\Delta t=\Delta\varphi=0} \sim 1 + \frac{1}{d_H^2}\left(\rho^2 + \frac{s^2}{2} - \rho\sqrt{\rho^2+s^2}\right) \\ &\sim 1 + \frac{s^4}{8(r^2-r_+^2)(r^2-r_-^2)} \qquad \text{for } \rho^2 >> s^2\,. \end{aligned} \tag{5.52}$$

Since it turns out that the divergences due to short distances come only from the term with $n = 0$ in the summation in (5.47), we focus on this term, which is given by

$$I \ \equiv \ \frac{\Omega_H}{8\pi l^3}\,\beta_H^2 \int_{r_+}^{\infty} d(r^2)\ \lim_{s\to 0}\ B \int_{A_0+B}^{\infty} dz \tag{5.53}$$

$$\times\ \frac{X^{1-\lambda}\,\{1+(\lambda-1)\log X\}}{\log^2 X\sqrt{z^2-1}} \frac{1}{\sqrt{(z-A_0)^2-B^2}} \frac{(z-A_0)+B}{(z-A_0+B)^2}\Big|_{\Delta\tau=\Delta\varphi=0}\,.$$

By introducing two new variables by

$$z' \ \equiv \ z - A_0\,, \qquad \delta \ = \ A_0 - 1 \sim \frac{1}{d_H^2}\left(\rho^2 + \frac{s^2}{2}\right)\,, \tag{5.54}$$

we get $\log X \sim \sqrt{2(z'+\delta)}$ up to $\mathcal{O}(z',\delta)$. Therefore we find the contribution to the integral from the region of the short distances,

$$
\begin{aligned}
I &\sim \frac{\Omega_H}{8\pi l^3}\,\beta_H^2 \int_{\epsilon_H^2} d(\rho^2) B \int_B dz' \frac{1+(\lambda-1)\sqrt{2(z'+\delta)}}{(z'+\delta)(z'+B)\sqrt{z'+\delta}\sqrt{z'^2-B^2}} \\
&= \frac{\Omega_H}{8\pi l^3}\,\beta_H^2 \int_{\epsilon_H^2} d(\rho^2)\; B \int^1 du \\
&\qquad \times \frac{1}{(1+(\delta/B)u)\,(1+u)\sqrt{1-u^2}} \left\{ \frac{1}{\sqrt{1+(\delta/B)\;u}} \left(\frac{u}{B}\right)^{3/2} + \sqrt{2}(\lambda-1)\frac{u}{B} \right\},
\end{aligned} \tag{5.55}
$$

where $u = B/z'$. We have regularized the integral by introducing the cutoff, ϵ_H, for the lower end of the integral as

$$
r_+^2 \quad\longrightarrow\quad r_+^2 + \epsilon_H^2\,. \tag{5.56}
$$

Thus, for $\epsilon_H, \rho \simeq s$ or $\epsilon_H, \rho >> s$, since $\delta/B \sim 1$ and $B \sim (\rho/d_H)^2$ hold, we obtain

$$
\begin{aligned}
I &\sim \Omega_H \beta_H^2 l^{-3} \int_{\epsilon_H^2} d(\rho^2) \left(B^{-3/2} + \; c\; B^{-1}\right) \\
&\sim \Omega_H \beta_H^2 l^{-3} \left(d_H^3 \frac{1}{\epsilon_H} + \; c' d_H^2 \log \epsilon_H^2\right),
\end{aligned} \tag{5.57}
$$

where c and c' are constants. On the other hand for $\epsilon_H, \rho << s$, since $\delta/B_0 \sim s/\rho$ holds, we obtain

$$
\begin{aligned}
I &\sim \Omega_H \beta_H^2 l^{-3} \int_{s^2} d(\rho^2) \left(\delta^{-3/2} + \; c\; \delta^{-1}\right) \\
&\sim \Omega_H \beta_H^2 l^{-3} \left(d_H^3 \frac{1}{s} + \; c' d_H^2 \log s^2\right).
\end{aligned} \tag{5.58}
$$

Therefore the divergences are given in terms of the larger cutoff, i.e., $\max\{\epsilon_H, s\}$.

Finally, we study divergences coming from the first term of (5.47), namely, $\beta_H F(\beta_H)$. As discussed in Sec 5.2 , this term has two sources of divergences. One is the integration over large r for $\lambda \le 1$. The other is the term with $n = 0$ in the summation in the integrand. This term becomes divergent for small s like

$$
\frac{1}{\sqrt{\sigma(x,x')}} \sim \frac{1}{s^2}\sqrt{(r^2-r_+^2)(r^2-r_-^2)}\,. \tag{5.59}
$$

For the case of $J = 0$, the analysis of divergent parts is performed in a similar way. Thus we do not give details of it, and just make some remarks. In this case, the term corresponding to $\beta_H F(\beta_H)$ and divergences associated with this term do not exist. The

entropy of the case $J = 0$ is obtained from the second term in (5.47) by replacing the factor $\Omega_H \beta_H$ with 2π. Thus divergences of the entropy are easily obtained by the same procedure.

Finally we make some comments on the entropies. First, from the expression of the entropies, we find that they contain various divergences coming from short distances such as $\epsilon_H^{-1}, \log \epsilon_H^2$, and s^{-2}. However not all of them are due to the existence of the outer horizon. We also find that the divergent terms are proportional to d_H. Hence they are proportional to the area of the outer horizon (r_+) for the $J = 0$ case, but this does not hold for the $J \neq 0$ case. Actually, the divergent terms vanish in the extreme limit $r_- \to r_+$. This agrees with the discussion (at the classical level) that extreme black holes have zero entropy [26, 27].

6 Conclusions and Discussions

We have investigated the thermodynamics of scalar fields on the three dimensional black holes in two approaches. One (approach I) is based on explicit mode expansion of the scalar fields and direct computation of the partition sum, and the other (approach II) is based on Hartle-Hawking Green functions. In both approaches, explicit expressions of the free energies and the entropies are obtained. We believe that we have provided reliable bases for the study of thermodynamics of scalar fields on the three dimensional black holes and that our results give useful insight for understanding of thermodynamics of black holes in four dimensions.

Our results also allow us to answer the interesting questions listed in the introduction at least for the three dimensional case.

First, we obtained physical quantities such as densities of states, free energies and entropies in approach I. Then we found that they crucially depend upon boundary conditions. In particular, divergent terms of the entropy are not necessarily due to the existence of the outer horizon. They also depend upon the boundary conditions. In addition, the entropy is not proportional to the area of the outer horizon, namely, the Bekenstein-Hawking formula is invalid.

Second, we constructed exact Hartle-Hawking Green functions on the three dimensional black holes. By making use of them, we obtained free energies and entropies. The divergent terms of the entropies, which come from short distances, are proportional to

$d_H = \sqrt{r_+^2 - r_-^2}$. Thus the Bekenstein- Hawking formula is not satisfied except for the $J = 0$ case. (For $J \neq 0$, there is another divergence which is the same as that of the free energies.) Furthermore, the divergences are not always due to the existence of the outer horizon and depend upon the regularization method. In addition, the results obtained in approach I and II are quite *different*.

Therefore we conclude that the Bekenstein-Hawking formula including the quantum scalar fields is not valid in general. Thus the relationship between the divergence of the entropies and the renormalization of the gravitational coupling constant is not clear. It is also obscure whether the divergences are due to the existence of the horizon.

The expressions of the entropies largely depend upon the method of calculation, boundary conditions and regularization, namely, upon its definition. Hence it is quite important to consider what kind of definition we should adopt. Without fixing the definition, we cannot discuss whether the entropies of scalar fields have a meaning as the number of states or whether they can be regarded as " geometric entropy ". Therefore we cannot understand the problem of the relationship between information loss and entropy of quantum fields until the problem of the definition is settled.

Fortunately, we have got explicit expressions of the thermodynamic quantities of scalar fields on the three dimensional black holes. Thus we think that it is possible to apply our results to various problems, for example, the problems discussed above, Hawking radiation [28] and the generalized second law [29]. Since the divergences appearing in our calculation can be absorbed into the renormalization of the *cosmological constant* , it may worth investigating the relationship between the divergence of the entropies and renormalization.

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Appendix

In this appendix, we summarize the derivation of the Feynman Green function in the universal covering space of three dimensional anti-de Sitter space (CAdS_3). Quantization of a scalar field in CAdS_n is discussed in [13]-[16], and the Feynman Green function is given [16] in terms of hypergeometric functions. In the three dimensional case, we find that the Feynman Green function is simplified and expressed in terms of elementary functions.

We parametrize CAdS_3 as follows

$$\begin{aligned} & -u^2 - v^2 + x^2 + y^2 = -l^{-2}\,, \\ & u = l\sin\tau\sec\rho\,, \qquad v = l\cos\tau\sec\rho\,, \\ & x = l\sin\theta\tan\rho\,, \qquad y = l\cos\theta\tan\rho\,, \end{aligned} \tag{A .1}$$

where $0 \le \rho < \pi/2\,,\quad 0 \le \theta < 2\pi\,,\quad -\infty\ < \tau < \infty$. Then the metric becomes

$$ds^2 \;=\; l^2\sec^2\rho\left(-\,d\tau^2 + d\rho^2 + \sin^2\rho\; d\theta^2\right)\,. \tag{A .2}$$

The field equation for a scalar field is given by

$$\left(\Box - \mu l^{-2}\right)\psi(x) \;=\; 0\,. \tag{A .3}$$

Making the separation of variables

$$\psi_{m\omega} \;=\; e^{-i\omega\tau}\; e^{im\theta}R(\rho)\,, \qquad (m \in \mathbf{Z})\,, \tag{A .4}$$

the equation for the radial function $R(\rho)$ is given as

$$\left(\partial_\rho^2 + \frac{1}{\sin\rho\;\cos\rho}\partial_\rho + \omega^2 - \frac{m^2}{\sin^2\rho} - \mu\sec^2\rho\right)R(\rho) \;=\; 0\,. \tag{A .5}$$

Let us make the change of variables $v = \sin^2\rho$, and define a function $f(v)$ by

$$R(\rho) \;=\; v^{|m|/2}(1-v)^{\lambda/2}f(v)\,, \tag{A .6}$$

with

$$\lambda \;=\; \lambda_\pm \equiv 1 \pm \sqrt{1+\mu}\,. \tag{A .7}$$

Then the radial equation above is reduced to the hypergeometric equation

$$\left[v(1-v)\partial_v^2 + \{c - (a+b+1)v\}\,\partial_v - ab\right]\; f(v) \;=\; 0\,, \tag{A .8}$$

where

$$\begin{aligned} a &= \frac{1}{2}(\lambda + |m| - \omega)\,, \\ b &= \frac{1}{2}(\lambda + |m| + \omega)\,, \\ c &= |m| + 1\,. \end{aligned} \tag{A .9}$$

If we require the regularity at $v = 0$, the solution is expressed by the Gauss' hypergeometric function F as

$$f(v) = F(a, b; c; v)\,. \tag{A .10}$$

Since $CAdS_3$ is not globally hyperbolic, it is necessary to impose boundary conditions at the spatial infinity. Following [13]-[15], we require the condition to conserve energy. This means that the surface integral of the energy-momentum tensor at the spatial infinity must vanish. This requirement leads to

$$|\omega| = \lambda + |m| + 2n \qquad (n = 0, 1, 2, ..,)\,, \tag{A .11}$$

where

$$\lambda = \begin{cases} \lambda_\pm & \text{for } 0 > \mu > -1\,, \\ \lambda_+ & \text{for } \mu \geq 0\,, \mu = -1 \end{cases}\,. \tag{A .12}$$

Then a becomes zero or a negative integer. Thus by using a mathematical formula [30] we get

$$\psi(x) = \sum_{m,n} [a_{mn}\psi_{mn} + (a_{mn}\psi_{mn})^*] \qquad (m \in \mathbf{Z}, \quad n = 0, 1, 2, ...)\,, \tag{A .13}$$

$$\psi_{mn} = C_{mn}\, e^{-i\omega\tau}\, e^{im\theta}\, (\sin\rho)^{|m|}\, (\cos\rho)^\lambda\, P_n^{(|m|, \lambda-1)}(\cos 2\rho)\,, \tag{A .14}$$

where $P_n^{(\alpha,\beta)}$ is a Jacobi Polynomial and C_{mn} is a normalization constant.

For the positive frequency part $\psi^{(+)}$ of the solution we can define a positive definite scalar product as

$$\left(\psi_1^{(+)}, \psi_2^{(+)}\right) \equiv -i \int_\Sigma d^2x \sqrt{-g} g^{0\nu} \psi_1^{(+)*} \overleftrightarrow{\partial}_\nu \psi_2^{(+)}\,, \tag{A .15}$$

where Σ is a spacelike hypersurface. Then the normalization constant C_{mn} is determined by the condition $\left(\psi_{mn}^{(+)}, \psi_{m'n'}^{(+)}\right) = \delta_{mm'}\delta_{nn'}$. By using the orthogonal relation with respect to Jacobi Polynomials [30],

$$\begin{aligned} &\int_0^{\pi/2} d\rho \tan\rho\, (\sin\rho)^{2|m|}\, (\cos\rho)^{2\lambda}\, P_n^{(|m|, \lambda-1)}(\cos 2\rho)\; P_{n'}^{(|m|, \lambda-1)}(\cos 2\rho) \\ = &\ \delta_{nn'} \frac{1}{2(2n+\lambda+|m|)} \frac{\Gamma(n+|m|+1)\Gamma(n+\lambda)}{n!\Gamma(n+\lambda+|m|)}\,, \end{aligned} \tag{A .16}$$

we get

$$C_{mn} \;=\; \left[\frac{n!\,\Gamma(\,|m|+\lambda+n)}{2\pi l(\,|m|+n)!\,\Gamma(\lambda+n)}\right]^{1/2} . \tag{A .17}$$

Now we quantize the scalar field by setting the commutation relation

$$\left[a_{mn}, a^{\dagger}_{m'n'}\right] = \delta_{mm'}\delta_{nn'} \,. \tag{A .18}$$

Then we get

$$\begin{aligned} \left[\psi(x),\ \psi(x')\right]_{\tau=\tau'} &= 0\,, \\ \left[\psi(x),\ \partial_{\tau'}\psi(x')\right]_{\tau=\tau'} &= -i\frac{1}{g^{\tau\tau}\sqrt{-g}}\delta(\theta-\theta')\delta(\rho-\rho')\,. \end{aligned} \tag{A .19}$$

Here we have used the orthogonal relation (A .16). The δ function is defined for the space of functions of the form as (A .14)

Let us define

$$\begin{aligned} -iG_F(x,x') &= \langle\, 0\,|\,T\,\{\psi(x)\psi(x')\}\,\,|\,0\,\rangle \\ &\equiv \theta(\tau-\tau')\sum_{m\,,n}\psi_{mn}(x)\psi^{*}_{mn}(x') + \ (x\leftrightarrow x')\,. \end{aligned} \tag{A .20}$$

From (A .19), it follows that

$$\left(\Box-\mu l^{-2}\right)G_F(x,x') \;=\; \frac{1}{\sqrt{-g}}\delta(x-x')\,, \tag{A .21}$$

namely G_F is the Feynman Green function.

Furthermore, we can perform the summation with respect m and n. First, we can set $x'=(\tau',\rho',\theta')=(0,0,0)$, (i.e. $(u',v',x',y')=(0,l,0,0)$) without loss of generality because CAdS_3 is homogeneous. Then only the term with $m=0$ contribute to the summation, i.e.,

$$-iG_F(x,0) \;=\; \frac{1}{2\pi l}\;e^{-i\lambda\,|\tau|}\,(\cos\rho)^{\lambda}\sum_{n=0}^{\infty}\;e^{-2in\,|\tau|}\;\;P_n^{(0,\lambda-1)}(\cos 2\rho)\,. \tag{A .22}$$

By making use of the mathematical formulae [31]

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{(\alpha+\beta+1)_k}{(\beta+1)_k}\;t^k\;P_k^{(\alpha,\beta)}(x) \\ &\quad= (1+t)^{-\alpha-\beta-1}F\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2};\beta+1;\frac{2t(x+1)}{(t+1)^2}\right)\,, \end{aligned} \tag{A .23}$$

we get

$$-iG_F(x,0) \equiv -iG_F(z) \;=\; \frac{l^{-1}}{2^{\lambda+1}\pi} z^{-\lambda}\; F\left(\frac{1}{2}\lambda, \frac{1}{2}(\lambda+1); \lambda; z^{-2}\right) . \qquad \text{(A .24)}$$

Here z is defined by

$$z \;=\; 1 + l^{-2}\sigma(x,0) + i\varepsilon\,, \qquad \text{(A .25)}$$

and $\sigma(x,x')$ is half of the distance between x and x' in the four dimensional embedding space, namely,

$$\sigma(x,x') \;=\; \frac{1}{2}\eta_{\mu\nu}(\xi-\xi')^{\mu}(\xi-\xi')^{\nu}\,, \qquad \text{(A .26)}$$

where $\eta_{\mu\nu} = diag(-1,-1,+1,+1)$ and ξ and ξ' are the coordinates in the embedding space. The infinitesimal imaginary part $i\varepsilon$ $(\varepsilon > 0)$ in z is added so that the Green function looks locally like the Minkowski one [13]. In the three dimensional case, from the mathematical formula,

$$F\left(a, \frac{1}{2}+a; 2a; z\right) \;=\; 2^{2a-1}(1-z)^{-1/2}\left[1+(1-z)^{1/2}\right]^{1-2a}\,, \qquad \text{(A .27)}$$

we find that the Feynman Green function is simplified to be

$$-iG_F(z) \;=\; \frac{l^{-1}}{4\pi}(z^2-1)^{-1/2}\left[z+(z^2-1)^{1/2}\right]^{1-\lambda} . \qquad \text{(A .28)}$$

This result is obtained also by replacing $|\tau|$ with $|\tau| - i\varepsilon$ so that $|e^{-2in|\tau|}| < 1$ holds and by utilizing the generating function of Jacobi Polynomials.

In the case with general x' , we have only to replace $\sigma(x,0)$ with $\sigma(x,x')$.

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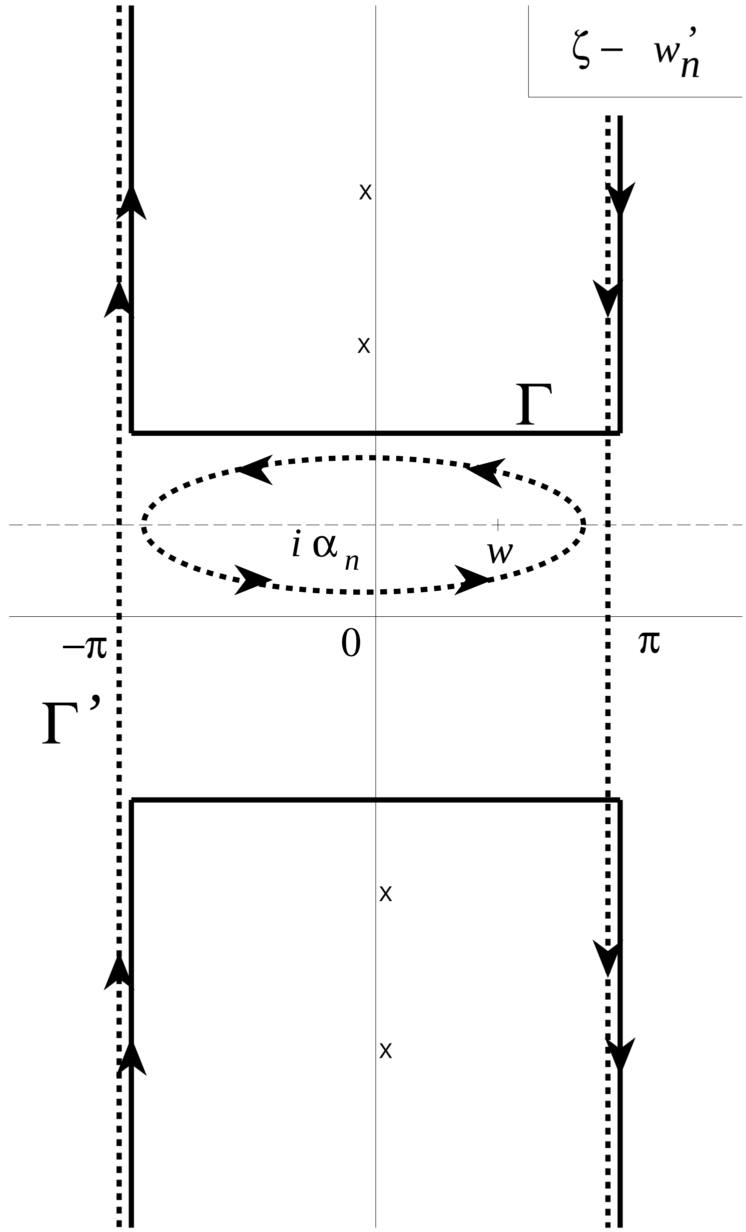

Fig. 1 : Contour Γ (solid line) and Contour Γ' (dashed line) in $\zeta - w'_n$ plane. The crosses ($\times$) indicate the singularities of $\tilde{G}^E_F(\zeta - w'_n)$ in the region $-\pi < \mathrm{Re}\ (\zeta - w'_n) \leq \pi$ for $r,\ r' \geq r_+$. $\alpha_n = r_- \Delta\phi^+_n / l$. In this figure, we show the contour Γ in the case of small $|\alpha_n|$. In the case of large and positive α_n, for example, the line $\zeta - w'_n = i\alpha_n$ is above the crosses, and we can not take a contour as Γ in this figure. In this case we have only to deform Γ maintaining the property that it can be deformed into a contour topologically equivalent to Γ'.